Exam 2 Review: Sections 2.1-2.5 and 3.1-3.4

Section 2.1. Consider a rabbit population satisfying the logistic equation

$$\frac{dP}{dt} = 2P - (0.005)P^2.$$

If the initial population is 120 rabbits, how many months does it take for $P(t)$ to reach 95% of its limiting population M?

$$\frac{dP}{dt} = 2P - 0.005P^2 = (0.005)P(400-P)$$

So $P(t) = \dfrac{120\cdot 400}{120 + 280e^{-2t}}$ ...... See Quiz 3 $\quad t = \dfrac{\ln\left(\dfrac{\frac{4800}{38} - 120}{280}\right)}{-2} \approx 1.896$ yrs.

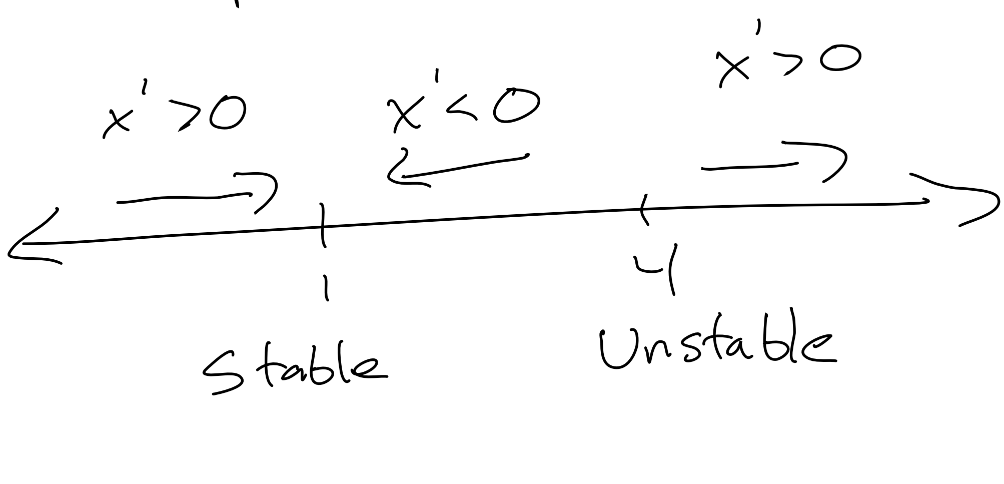
Section 2.2 Draw the phase diagram for the autonomous differential equation

$$\frac{dx}{dt} = x^2 - 5x + 4$$

and determine which critical points are stable and unstable.

$0 = x^2 - 5x + 4 = (x-4)(x-1)$.

Critical pts at $x = 1, 4$



Section 2.3 Consider a body that moves horizontally through a medium whose resistance is proportional to the square of velocity so that

$$\frac{dv}{dt} = -2v^2.$$

Assuming that $v(0) = 1$ and $x(0) = 1$, find the position $x(t)$ as a function of t.

$$-\frac{1}{v} = \int \frac{dv}{v^2} = \int -2\,dt = -2t + C$$

$$-1 = -\frac{1}{v(0)} = C$$

Thus $-\frac{1}{v} = -2t - 1$

$$v = \frac{1}{2t+1}$$

and $x = \int \frac{1}{2t+1}\,dt = \frac{1}{2} \ln(2t+1) + C$

$x(0) = 1 = C$, so

$$x(t) = 1 + \frac{1}{2}\ln(2t+1).$$

Sections 2.4 Use the Euler method to find an approximation for $y(2)$ using a step size of $h = 0.5$ for the differential equation

$$yy' = 2x^3, \quad y(1) = 3.$$

$y' = \frac{2x^3}{y}$, $y(1) = 3$. So $(x_0, y_0) = (1, 3)$

$x_1 = 1.5$, $y_1 = 3 + \frac{1}{2} \cdot y'(1, 3) = \frac{10}{3}$

$x_2 = 2$, $y_2 = \frac{10}{3} + \frac{1}{2} \cdot y'\left(1.5, \frac{10}{3}\right) = \frac{1043}{240}$.

Thus $y(2) \approx \frac{1043}{240}$.

Section 2.5 Use the Improved Euler method to find an approximation for $y(2)$ using a step size of $h = 0.5$ for the differential equation

$$yy' = 2x^3, \quad y(1) = 3.$$

$y' = \frac{2x^3}{y}$, $y(1) = 3$. So $(x_0, y_0) = (1, 3)$.

$x_1 = 1.5$, $k_1 = y'(1,3) = \frac{2}{3}$ and $u_1 = 3 + \frac{1}{2} k_1 = \frac{10}{3}$

$k_2 = y'(1.5, \frac{10}{3}) = \frac{81}{40}$.

Then $y_1 = 3 + \frac{1}{2} \cdot \frac{1}{2}\left(\frac{2}{3} + \frac{81}{40}\right) \approx 3.67$.

$x_2 = 2$, $k_1 = y'(1.5, 3.67) \approx 1.84$ and $u_2 = 3.67 + \frac{1}{2} \cdot 1.84 = 4.59$

$k_2 = y'(2, 4.59) \approx 3.49$.

Then $y_2 = 3.67 + \frac{1}{2} \cdot \frac{1}{2}(1.84 + 3.49) \approx 5$.

So $y(2) \approx 5$.

Sections 3.1-3.3 Find the general form of they solution to the differential equation

$$6y^{(4)} + 5y^{(3)} + 25y'' + 20y' + 4 = 0$$

which has characteristic function

$$(r^2 + 4)(6r^2 + 5r + 1) = 0.$$

$r = \pm 2i$

$(3r+1)(2r+1) = 0$

$r = -\frac{1}{3}, -\frac{1}{2}$

Then $y = c_1 e^{-x/3} + c_2 e^{-x/2} + (a_1 \cos 2x + b_1 \sin 2x).$

Section 3.4 A 12-lb weight (mass m=0.275 slugs) is attached both to a vertically suspended spring that it stretches 6 in. and to a dashpot that provides 3 lb of resistance for every foot per second of velocity.

(a) The weight is pulled down 1 ft below its static equilibrium position and then released from rest at time $t = 0$, find its position function $x(t)$.

(b) Determine if the motion is over-damped, critically damped or under-damped.

$$mx'' + cx' + kx = 0$$

Use the formula

$Lbs - K \cdot S_0 = 0$ to get

$12 - K \cdot \frac{1}{2} = 0 \Rightarrow K = 24.$

Also $c = 3.$

So we have $0.375x'' + 3x' + 24x = 0$

or $x'' + 8x' + 64x = 0$

Char. Eqn $\equiv$ $r^2 + 8r + 64 = 0 \Rightarrow r = -4 \pm i\sqrt{48}.$

So $x(t) = e^{-4t}(A\cos\sqrt{48}t + B\sin\sqrt{48}t)$ and the motion is <u>under</u>-<u>damped</u>.

We also have $x(0) = -1$ and $v(0) = 0$.

So $x(0) = -1 = A$ and

$$v(0) = 0 = -4e^{-4t}(-\cos\sqrt{48}t + B\sin\sqrt{48}t) + e^{-4t}(\sqrt{48}\sin\sqrt{48}t + \sqrt{48}B \times \cos\sqrt{48}t) \text{ @ } 0$$

$$= 4 + \sqrt{48}B \Rightarrow B = \frac{-4}{\sqrt{48}} = -1/\sqrt{3}$$

Thus $x(t) = -e^{-4t}\left(\cos\sqrt{48}t + \frac{1}{\sqrt{3}}\sin\sqrt{48}t\right).$